

# 3+1 formulation of non-ideal hydrodynamics

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Accepted for publication in the MNRAS

## ABSTRACT

The equations governing dissipative relativistic hydrodynamics are formulated within the 3+1 approach for arbitrary spacetimes. Dissipation is accounted for by applying the theory of extended causal thermodynamics (Israel-Stewart theory). This description eliminates the causality violating infinite signal speeds present in the conventional Navier-Stokes equation. As an example we treat the astrophysically relevant case of stationary and axisymmetric spacetimes, including the Kerr metric. The equations take a simpler form whenever the inertia due to the dissipative contribution can be neglected.

**Key words:** relativity – hydrodynamics – black hole physics – accretion, accretion discs – galaxies: active – stars: neutron

## 1 INTRODUCTION

The motion of dissipative fluids in strong gravitational fields is of considerable interest in various fields of astrophysics and cosmology. Examples include accretion discs around compact objects, rotating relativistic fluid configurations such as supermassive stars, neutron stars or strange (boson) stars, the collapse of stellar objects and the merging of compact objects. Examples in cosmology cover inflationary cosmological scenarios and the evolution of density fluctuations.

The non-stationary modeling of relativistic matter is most conveniently performed within

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the 3+1 formalism, where the equations of motion for gravitational- and matter fields are decomposed w.r.t. a congruence of fiducial observers (FIDOs), allowing to express time derivatives on a per-unit-universal-time basis. The 3+1 representation of the equations for ideal, non-dissipative (relativistic) hydrodynamics as well as Maxwell's equations for the electric and magnetic fields and Einstein's equations for the gravitational fields have been discussed by many authors, both in a cosmological context (e.g. Durrer & Straumann 1988) and in the case of black hole spacetimes (e.g. Thorne & Macdonald 1982).

Modeling dissipative processes requires non-equilibrium or irreversible thermodynamics. Standard (or classical), irreversible thermodynamics (in the following referred to as *standard thermodynamics*) was first extended from Newtonian to relativistic fluids by Eckart (1940). However, the Eckart theory, and a variation thereof by Landau & Lifshitz (1959) shares with its Newtonian counterpart serious problems. Notably that dissipative fluctuations propagate at an infinite speed. In addition, generic short wavelength secular instabilities driven by dissipative processes exist (Lindblom & Hiscock 1983) and finally, no well-posed initial value problem exists for rotating fluid configurations.

At the origin of these problems in standard thermodynamics is the description of non-equilibrium via the local equilibrium states alone, i.e. it is assumed that local thermodynamic equilibrium is established on an infinitely short time-scale (see e.g. Jou, Casas-Vázquez & Lebon 1997 for an introduction). In extended theories of irreversible thermodynamics the set of thermodynamic variables is extended to include the dissipative variables. This restores causality and stability under a wide range of conditions (Hiscock & Lindblom 1983). A non-relativistic extended theory was proposed by Müller (1967), and was then generalized to the relativistic case by Israel (1976) and Stewart (1977). The extended theory is commonly referred to as *causal thermodynamics*, second-order thermodynamics or transient thermodynamics.

The problem of non-causality has recently received attention in the context of transonic accretion discs. Two different approaches have been proposed to overcome this difficulty. For steady flow Narayan (1992) has established causality by calculating the coefficient of kinematic viscosity within an extended version of flux limited diffusion theory (Levermore & Pomraning 1981), assuming a particular steady state phase-space distribution function for the turbulent fluid elements in the disc. The influence of this modified viscosity coefficient was studied in stationary accretion discs by Popham & Narayan (1994) and Syer & Narayan (1993). A relativistic generalization of the modified viscosity has been proposed, and used in

models of stationary relativistic accretion discs (Peitz & Appl (1997)). A different approach by Papaloizou & Szuszkiewicz (1994), which is related to a causal description for the thermodynamics, was used by Kley & Papaloizou (1997) in time-dependent models for accretion disc boundary layers. Gammie & Popham (1997) have recently considered a similar extension to stationary relativistic accretion discs. In cosmology the theory of causal thermodynamics is currently attracting growing interest predominantly in the contexts of re-heating processes after inflation (Zimdahl, Pavón & Maartens 1997) and in linear perturbation theory for the evolution of density fluctuations (Maartens & Triginer 1997).

This paper provides a complete set of equations for dissipative fluid mechanics in their 3+1 representation, using a causal description of thermodynamics. In Section 2 the basic elements of both standard and causal dissipative hydrodynamics are reviewed in their spacetime description, and their 3+1 representation is derived in Section 3. As a particular application of astrophysical interest we specify the system of equations given in Section 3 to the case of a stationary, axisymmetric back ground in Section 4.

## 2 DISSIPATIVE RELATIVISTIC HYDRODYNAMICS

The equations of ideal relativistic fluid mechanics and the equations for dissipative relativistic fluid mechanics in both the standard irreversible and the extended causal thermodynamics description are reviewed. For a detailed discussion see Israel & Stewart (1979), Hiscock & Lindblom (1983) or a recent treatment by Maartens (1997).

### 2.1 Notation

We use geometrized units such that  $c = 1 = G$ . Tensor fields defined on spacetime  $(\mathcal{M}, g)$  with metric  $g$  of signature  $(-, +, +, +)$  appear in roman (e.g.  $u, T$ ), while scalar functions are in italic. The velocity  $u$  of the fluid is normalized to  $u \cdot u = -1$ . The tensor  $h = g + u \otimes u$  projects into the 3-space orthogonal to  $u$ , the local rest frame of the fluid (LRF). Total projections (i.e. projection in any free index) parallel to and orthogonal to  $u$  are denoted by  $(\ )_u$  and  $(\ )_h$ . If  $A$ ,  $B$  and  $C$  are a scalar, vector and rank-2 tensor field on  $(\mathcal{M}, g)$ , respectively, then

$$\begin{aligned} A_u &= A_h = A, \\ B_u &= B_u = u \cdot B, \quad B_h = h \cdot B, \\ C_u &= C_u = u \cdot C \cdot u, \quad C_h = h \cdot C \cdot h. \end{aligned} \tag{1}$$

Two covariant differential operators  $\dot{D}$  and  $D$  are defined as projections of the affine connection  $\nabla$  on  $(\mathcal{M}, g)$  into directions parallel and orthogonal to  $u$ ,

$$\dot{D} \equiv \nabla_u = u \cdot \nabla, \quad (2)$$

$$D \equiv \nabla_h = h \cdot \nabla. \quad (3)$$

For 2-tensor fields  $C$  we further introduce (anti-) symmetrization operators  $\langle \rangle^a$  and  $\langle \rangle^s$  by  $C = \langle C \rangle^a$  if  $C$  is anti-symmetric and  $C = \langle C \rangle^s$  if  $C$  is symmetric and trace-free. Finally, the irreducible decomposition of  $\nabla u$  yields

$$\nabla u = \sigma + \omega + 1/3\Theta h - a \otimes u, \quad (4)$$

with shear  $\sigma \equiv \langle \nabla u \rangle_h^s$ , vorticity  $\omega \equiv \langle \nabla u \rangle_h^a$ , acceleration  $a \equiv \dot{D}u$  and expansion  $\Theta \equiv D \cdot u = \nabla \cdot u$  of the fluid trajectories.

## 2.2 Perfect fluids

A perfect fluid is described by the velocity  $u$ , baryon number density  $n$ , mass-energy density  $\rho$ , isotropic pressure  $p$  and specific entropy  $s$ , which are subject to the conservation laws

$$0 = \nabla \cdot n, \quad (5)$$

$$0 = \nabla \cdot T. \quad (6)$$

The particle current vector  $n$  and the symmetric stress-energy tensor  $T$  are given by

$$n = nu, \quad (7)$$

$$T = \rho u \otimes u + ph. \quad (8)$$

The LRF conservation laws for energy and momentum result from projecting (6) parallel and orthogonal to  $u$ . With (8) one can write  $0 = (\nabla \cdot T)_u$  and  $0 = (\nabla \cdot T)_h$  as

$$0 = \dot{D}\rho + (\rho + p)\Theta, \quad (9)$$

$$0 = Dp + (\rho + p)a. \quad (10)$$

The metric  $g$  is coupled to stress-energy  $T$  by Einstein's equations

$$G = 8\pi T, \quad (11)$$

where  $G$  is the Einstein tensor.

Thermodynamic scalar functions are defined in the LRF. The entropy flux

$$s = sn \quad (12)$$

is conserved along flow lines (adiabatic flow),

$$0 = \nabla \cdot \mathbf{s} . \quad (13)$$

The temperature  $T$  is defined via the Gibbs equation

$$Tds = d(\rho/n) + pd(1/n) , \quad (14)$$

where  $d$  is the exterior derivative on  $(\mathcal{M}, g)$ . In general, two thermodynamic scalars are needed as independent variables, which we choose  $n$  and  $\rho$ . A scalar equation of state, e.g.  $p = p(n, \rho)$ , closes the system of equations (5), (9)-(11) for the dynamical variables  $\{n, \rho, \mathbf{u}, g\}$ .

### 2.3 Dissipative fluids

Choosing the particle current for dissipative fluids by (7) corresponds to selecting an average velocity  $\mathbf{u}$  such that the particle flux in the associated rest frame vanishes. This so-called particle frame or Eckart frame (Eckart 1940) is the natural frame in systems where particle number is conserved (see Israel & Stewart 1979 for the alternative energy frame description). The state of the fluid is assumed close to a fictitious thermodynamic equilibrium state, characterized by the local thermodynamic equilibrium scalars  $n_0, \rho_0, p_0, s_0, T_0$  and the local equilibrium velocity  $\mathbf{u}_0$ , which in the Eckart frame can be chosen such that only the pressure  $p$  deviates from the local equilibrium pressure  $p_0$  by the bulk viscous pressure  $\Pi = p - p_0$ , whereas  $n = n_0$  and  $\rho = \rho_0$ . Dropping subscripts 0 allows then to write the general stress-energy tensor for dissipative fluids as

$$\mathbf{T} = \rho \mathbf{u} \otimes \mathbf{u} + (p + \Pi) \mathbf{h} + \mathbf{q} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{q} + \boldsymbol{\pi} , \quad (15)$$

where the heat flux  $\mathbf{q}$  relative to the particle frame and the anisotropic stress tensor  $\boldsymbol{\pi}$  are orthogonal to  $\mathbf{u}$ ,

$$\mathbf{q} = \mathbf{q}_h , \quad \boldsymbol{\pi} = \langle \boldsymbol{\pi} \rangle_h^s . \quad (16)$$

Conservation laws again hold for  $n$  and  $T$ . However, in irreversible thermodynamics the entropy is no longer conserved. According to the second law, the rate of entropy generation must therefore be positive definite,

$$\nabla \cdot \mathbf{s} \geq 0 , \quad (17)$$

implying that  $\mathbf{s}$  has a dissipative vector contribution  $\mathbf{R}$  in excess to (12),

$$\mathbf{s} = \mathbf{s}_n + \mathbf{R}/T . \quad (18)$$

Following the phenomenological Israel-Stewart approach,  $s = s_0$  remains related to  $T = T_0$  by the Gibbs equation (14). The dissipative part  $R$  is assumed to be an algebraic function of  $n$  and  $T$  only, which vanishes in equilibrium ( $R_0 = 0$ ). The theories of standard irreversible thermodynamics and of extended causal thermodynamics differ in the forms of  $R$ , as given below.

The equations of energy and momentum conservation for (15) can be written as

$$0 = \dot{D}\rho + (\rho + p + \Pi)\Theta + \sigma : \pi + (2a + D) \cdot q, \quad (19)$$

$$0 = (\rho + p + \Pi)a + D(p + \Pi) + D \cdot \pi + \pi \cdot a + (\dot{D}q)_h + (\sigma + \omega + 4/3\Theta h) \cdot q \quad (20)$$

where  $B : C \equiv \text{tr}(B \cdot C)$  for 2-tensor fields  $B, C$ . The above treatment applies for a single-component fluid and allows a natural extension to multi-component fluids.

## 2.4 Standard thermodynamics

Standard thermodynamics assumes a linear dependence of  $R$  on the *thermodynamic fluxes*  $\{\Pi, q, \pi\}$ , which is possible only if (18) takes the form

$$s = s_n + q/T. \quad (21)$$

Using (5), (19) and (20) yields the entropy generation rate

$$T\nabla \cdot s = -\Theta\Pi - (D \ln T + a) \cdot q - \sigma : \pi. \quad (22)$$

The simplest relation to be imposed between the thermodynamic fluxes  $\{\Pi, q, \pi\}$  and the *thermodynamic forces*  $\{3\Theta, D \ln T + a, \sigma\}$  in agreement with (17) is linear,

$$\Pi = -\zeta\Theta, \quad (23)$$

$$q = -\lambda T(D \ln T + a), \quad (24)$$

$$\pi = -2\eta\sigma, \quad (25)$$

with non-negative coefficients of bulk viscosity  $\zeta(\rho, n)$ , thermal conductivity,  $\lambda(\rho, n)$  and shear viscosity  $\eta(\rho, n)$ . This brings (22) to

$$T\nabla \cdot s = \Pi^2/\zeta + q \cdot q/(\lambda T) + \pi : \pi/(2\eta), \quad (26)$$

which on using (5), (14), (19) and (20) yields an evolution equation for the entropy density  $s$ ,

$$Tn\dot{D}s = -\Theta\Pi - \nabla \cdot q - a \cdot q - \pi : \sigma. \quad (27)$$

As a consequence of (23)-(25), the thermodynamic fluxes react instantaneously to the corresponding thermodynamic forces, implying propagation of signals at (causality violating) infinite speed. In standard thermodynamics the dynamics of the fluid is governed by the (compressible) Navier-Stokes equation, which results from substitution of (23)-(25) into (20).

## 2.5 Causal thermodynamics

Kinetic theory can motivate that  $R$  is second-order in the dissipative terms (Israel & Stewart 1979). Truncation at first order removes terms necessary for causality and stability. The most general algebraic form for  $R$  of at most second-order in the dissipative fluxes leads to

$$s = s_n + q/T - 1/2 (\beta_0 \Pi^2 + \beta_1 q \cdot q + \beta_2 \pi : \pi) u/T + \alpha_0 \Pi q/T + \alpha_1 \pi \cdot q/T . \quad (28)$$

The entropy density measured in LRF then becomes

$$-s \cdot u = s_n - 1/2 (\beta_0 \Pi^2 + \beta_1 q \cdot q + \beta_2 \pi : \pi) . \quad (29)$$

The negative sign of the non-equilibrium contributions reflects the fact that the entropy density is maximum in equilibrium. The thermodynamic coefficients  $\beta_j(\rho, n) \geq 0$  in (28) model deviations of the physical entropy density from  $s_n$  due to scalar/vector/tensor dissipative contributions to  $R$ . The  $\alpha_i(\rho, n)$  model contributions due to viscous/heat coupling, which do not influence the physical entropy density (29).

The entropy generation rate associated with (28) follows from (5), (14), (19) and (20) to

$$T \nabla \cdot s = -\Pi X - q \cdot Y - \pi : \langle Z \rangle^s \quad (30)$$

with the scalar, vector and rank-2 tensor fields

$$X = \Theta + \beta_0 \dot{\Pi} - \alpha_0 D \cdot q - \kappa_0 T q \cdot \nabla (\alpha_0/T) + 1/2 \Pi T \nabla \cdot (\beta_0 u/T) , \quad (31)$$

$$Y = \nabla \ln T + a + \beta_1 \dot{q} - \alpha_0 \nabla \Pi - \alpha_1 \nabla \cdot \pi - (1 - \kappa_0) \Pi T \nabla (\alpha_0/T) - (1 - \kappa_1) T \pi \cdot \nabla (\alpha_1/T) + 1/2 T q \nabla \cdot (\beta_1 u/T) , \quad (32)$$

$$Z = \nabla u + \beta_2 \dot{\pi} - \alpha_1 \nabla q - \kappa_1 T q \otimes \nabla (\alpha_1/T) + 1/2 T \pi \nabla \cdot (\beta_2 u/T) . \quad (33)$$

The simplest *evolution equations* for the *causal thermodynamic fluxes*  $\{\Pi, q, \pi\}$  in agreement with the second law (17) are again linear relationships,

$$\Pi = -\zeta X , \quad (34)$$

$$q = -\lambda T Y_h , \quad (35)$$

$$\pi = -2\eta \langle Z \rangle_h^s . \quad (36)$$

Two additional thermodynamic coefficients  $\kappa_k(\rho, n)$  had to be introduced in (31)-(33) as a consequence of the ambiguity involved in factoring terms which involve products  $\Pi q$  and  $\pi \cdot q$  in (28). Furthermore, (34)-(36) contain terms involving gradients of the  $\alpha_i$  and  $\beta_j$ . Since the  $\kappa_k$  are unknown a priori, these terms could be important even if the gradients themselves are small (Hiscock & Lindblom 1983). Finally, in (31)-(33) we neglected further contributions due to additional coupling terms between  $\{\Pi, q, \pi\}$  and  $a, \omega$ , which can be shown to exist in kinetic theory (Israel & Stewart 1979).

The complexity of the full evolution equations (34)-(36) makes applications tractable only if certain simplifications are made. A particularly simple set of evolution equations results from the assumptions (Maartens 1997)

$$0 = \kappa_0 = \kappa_1, \quad (37)$$

$$0 = \alpha_0 = \alpha_1, \quad (38)$$

$$0 \simeq \nabla \cdot (\beta_j u / T), \quad (39)$$

where (37) reflects essentially the lack of knowledge on the  $\kappa_k$  while (38) neglects the coupling between heat flux and viscosity. Implications of (39) are multifold and need to be justified after a particular solution was found using a parametrization for  $\beta_j(\rho, n)$ . The evolution equations resulting from (34)-(36) under the assumptions (37)-(39) are of covariant relativistic Maxwell-Cattaneo form,

$$\tau_0(\dot{D}\Pi)_h + \Pi = \tilde{\Pi}, \quad (40)$$

$$\tau_1(\dot{D}q)_h + q = \tilde{q}, \quad (41)$$

$$\tau_2(\dot{D}\pi)_h + \pi = \tilde{\pi}, \quad (42)$$

with the relaxation times  $\tau_j(\rho, n)$  given by

$$\tau_0 = \zeta\beta_0, \quad \tau_1 = \lambda T\beta_1, \quad \tau_2 = 2\eta\beta_2, \quad (43)$$

and  $\{\tilde{\Pi}, \tilde{q}, \tilde{\pi}\}$  the (re-named) standard thermodynamic fluxes as in (23)-(25). In contrast to the algebraic constraint equations (23)-(25), the evolution equations (40)-(42) are first order partial differential equations, which assure that in the LRF the viscous bulk/shear stresses and the heat flux relax towards their standard limits  $\{\tilde{\Pi}, \tilde{q}, \tilde{\pi}\}$  on time-scales  $\tau_j$ . The relaxation times  $\tau_j$  follow in principle from kinetic theory, but can be estimated as mean collision times,  $1/\tau \sim n\Sigma v$ , with  $\Sigma$  the collision cross section and  $v$  the mean particle speed.



For later use we re-write (40)-(42) as

$$\dot{\Pi} = \frac{1}{\tau_0} (\tilde{\Pi} - \Pi) , \quad (44)$$

$$\dot{\mathbf{q}} = \frac{1}{\tau_1} (\tilde{\mathbf{q}} - \mathbf{q}) + (\mathbf{q} \cdot \mathbf{a}) \mathbf{u} , \quad (45)$$

$$\dot{\boldsymbol{\pi}} = \frac{1}{\tau_2} (\tilde{\boldsymbol{\pi}} - \boldsymbol{\pi}) + \mathbf{u} \otimes (\boldsymbol{\pi} \cdot \mathbf{a}) + (\boldsymbol{\pi} \cdot \mathbf{a}) \otimes \mathbf{u} . \quad (46)$$

The conservation laws (5), (19), (20) and the Einstein equations (11), together with the evolution equations (34)-(36) constitute a complete system of hyperbolic first order PDEs for the solution vector of 24 ( $=1+1+3+10+1+3+5$ ) dynamical variables  $\{n, \rho, \mathbf{u}, \mathbf{g}, \Pi, \mathbf{q}, \boldsymbol{\pi}\}$ . This system represents a causal and stable theory for dissipative fluids (Hiscock & Lindblom 1983).

## 2.6 Weakly dissipative fluids

In many applications the inertia due to the dissipative contributions  $\{\Pi, \mathbf{q}, \boldsymbol{\pi}\}$  can be neglected. In addition, it is convenient to simplify the evolution equations (40)-(42), which depend on the kinematic properties  $\{\Theta, \mathbf{a}, \boldsymbol{\sigma}\}$  of the fluid, among which  $\boldsymbol{\sigma}$  is particularly expensive to compute. An appropriate simplification can be obtained by calculating (40)-(42) under the assumption of vanishing acceleration. Thermodynamic fluxes and thermodynamic forces calculated in this limit are underlined in the following. Note that the assumption of geodesic trajectories is only made for the calculation of the dissipative terms and not for the dynamics, i.e. we do not assume a geodesic velocity field satisfying  $\nabla \mathbf{u} \mathbf{u} = 0$ , but rather leave  $\mathbf{u}$  unspecified. In the following this will be referred to as the weakly dissipative limit. In this limit the standard constraint equations (23)-(25) reduce to

$$\underline{\tilde{\Pi}} = -\zeta \underline{\Theta} , \quad (47)$$

$$\underline{\tilde{\mathbf{q}}} = -\lambda T \underline{\mathbf{D}} \ln T , \quad (48)$$

$$\underline{\tilde{\boldsymbol{\pi}}} = -2\eta \underline{\boldsymbol{\sigma}} , \quad (49)$$

with the thermodynamic forces calculated from the kinematic properties

$$\underline{\Theta} \equiv \Theta|_{\mathbf{a}=0} , \quad \underline{\mathbf{a}} \equiv \mathbf{a}|_{\mathbf{a}=0} = 0 , \quad \underline{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma}|_{\mathbf{a}=0} = \langle \nabla \mathbf{u} \rangle^s - 1/3 \underline{\Theta} \mathbf{h} . \quad (50)$$

The causal evolution equations (44)-(46) simplify to

$$\dot{\underline{\Pi}} = \frac{1}{\tau_0} (\underline{\tilde{\Pi}} - \underline{\Pi}) , \quad (51)$$

$$\dot{\underline{\mathbf{q}}} = \frac{1}{\tau_1} (\underline{\tilde{\mathbf{q}}} - \underline{\mathbf{q}}) , \quad (52)$$

$$\dot{\underline{\pi}} = \frac{1}{\tau_2} (\tilde{\underline{\pi}} - \underline{\pi}) . \quad (53)$$

### 3 DISSIPATIVE HYDRODYNAMICS IN 3+1 FORMULATION

Particularly useful for time-dependent calculations in general relativity is the 3+1 formulation, where time derivatives are always with respect to globally defined universal time. Applications of 3+1 hydrodynamics and 3+1 magnetohydrodynamics have been mostly restricted to ideal fluids (see Bonazzola et al. 1993 for an exception). A collection of numerous general relativistic equations in the 3+1 representation can be found in Durrer & Straumann (1988). We give a 3+1 representation of relativistic dissipative hydrodynamics for both, standard and causal thermodynamics. For a detailed derivation of the equations presented in this section we refer to Peitz (Peitz 1998).

#### 3.1 Generalities on the 3+1 formalism

Assuming that spacetime  $(\mathcal{M}, g)$  admits a slicing by slices  $\Sigma_t$ , i.e. there is a diffeomorphism  $\Phi : \mathcal{M} \mapsto \Sigma \times I, I \subset \mathbb{R}$ , such that the manifolds  $\Sigma_t = \Phi^{-1}(\Sigma \times \{t\})$  are spacelike and the curves  $\Phi^{-1}(m, t)$  are timelike. These curves define a vector field  $\partial_t$  which can be decomposed into normal and parallel components relative to the slicing,

$$\partial_t = \alpha \hat{n} + \boldsymbol{\beta} . \quad (54)$$

Here  $\hat{n}$  is the timelike unit normal field (congruence of fiducial observers=FIDOs) and  $\boldsymbol{\beta}$  is tangent to the slices  $\Sigma_t$ .  $\alpha$  is the lapse function and  $\boldsymbol{\beta}$  is the shift vector field. A coordinate system  $\{x^i\}$  on  $\Sigma$  induces natural coordinates on  $\mathcal{M}$ , i.e.  $\Phi^{-1}(m, t)$  has coordinates  $(t, x^i)$  if  $m \in \Sigma$  has coordinates  $x^i$ . The timelike curves  $\partial_t$  have constant spatial coordinates (preferred timelike curves). Now set  $\boldsymbol{\beta} = \beta^i \partial_i$  (where  $\partial_i \equiv \partial/\partial x^i$ ). From  $g(\hat{n}, \partial_i) = 0$  one finds  $g(\partial_t, \partial_t) = -(\alpha^2 - \beta^i \beta_i)$  and  $g(\partial_t, \partial_i) = \beta_i$ . In coordinates co-moving with the FIDOs the metric thus reads

$$g = -(\alpha^2 - \beta^i \beta_i) dt \otimes dt + \beta_i dt \otimes dx^i + \beta_i dx^i \otimes dt + \gamma_{ij} dx^i \otimes dx^j \quad (55)$$

$$= -\alpha^2 dt \otimes dt + \gamma_{ij} (dx^i + \beta^i dt) \otimes (dx^j + \beta^j dt) . \quad (56)$$

The forms  $dt$  and  $dx^i + \beta^i dt$  are thus orthogonal.  $\boldsymbol{\gamma}$  is the metric induced on  $\Sigma_t$ , and the affine connection on  $(\Sigma_t, \boldsymbol{\gamma})$  is denoted by  $\boldsymbol{\nabla}$ .

The tangent and cotangent spaces of  $\mathcal{M}$  have two natural decompositions, which give rise to two types of bases of vector fields and 1-forms. These are the dual pair  $\{\partial_\mu\}$  and

$\{dx^\mu\}$  for comoving coordinates  $\{x^\mu\}$  and, on the other hand, the dual pair  $\{\hat{n}, \partial_i\}$  and  $\{\alpha dt, dx^i + \beta^i dt\}$ . Instead of the coordinate basis  $\{\partial_i\}$  one may also use an orthonormal horizontal basis  $\{\mathbf{e}_i\}$  with  $g(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$ , together with the dual basis  $\{\boldsymbol{\vartheta}^j\}$  for  $\{dx^i\}$ . Then one has the two dual pairs  $\{\partial_t, \mathbf{e}_i\}$ ,  $\{dt, \boldsymbol{\vartheta}^i\}$  and  $\{e_0 = \hat{n}, \mathbf{e}_i\}$ ,  $\{\theta^\mu\}$  with the orthonormal tetrad  $\{\theta^\mu\}$  given by  $\theta^0 = \alpha dt$  and  $\theta^i = \boldsymbol{\vartheta}^i + \beta^i dt$ , with  $\beta^i$  defined by  $\boldsymbol{\beta} = \beta^i \mathbf{e}_i$ . From (54) follows the relation

$$e_0 = \hat{n} = \frac{1}{\alpha} (\partial_t - \boldsymbol{\beta}) . \quad (57)$$

The 3+1 representation of respectively a scalar field  $A$ , a vector field  $B$  and a symmetric rank-2 tensor field  $C$  defined on  $(\mathcal{M}, g)$  is understood by a representation with respect to the basis  $\{e_0 = \hat{n}, \mathbf{e}_i\}$ , which we shall write in the following as

$$A = E_A , \quad (58)$$

$$B = E_B e_0 + \mathbf{S}_B , \quad \mathbf{S}_B = S_B^i \mathbf{e}_i , \quad (59)$$

$$C = E_C e_0 \otimes e_0 + \mathbf{S}_C \otimes e_0 + e_0 \otimes \mathbf{S}_C + \mathbf{T}_C , \quad \mathbf{S}_C = S_C^i \mathbf{e}_i , \quad \mathbf{T}_C = T_C^{ij} \mathbf{e}_i \otimes \mathbf{e}_j . \quad (60)$$

The vector field  $\mathbf{S}_B$  on  $(\Sigma_t, \boldsymbol{\gamma})$  corresponding to the vector field  $B$  on  $(\mathcal{M}, g)$  is referred to as the horizontal part of  $B$  and, accordingly, the tensor field  $\mathbf{T}_C$  on  $(\Sigma_t, \boldsymbol{\gamma})$  is the horizontal part of  $C$ . Horizontal tensor fields appear bold face. They can be viewed as spatial components of the fields on  $(\mathcal{M}, g)$  (which appear as a subscript), after having been projected onto the 3-space orthogonal to  $\hat{n}$  by  $(\cdot)_\gamma$ . If  $B = \mathbf{S}_B$ ,  $B$  is said to be a spatial vector field and, respectively,  $C$  is a spatial tensor field if  $C = \mathbf{T}_C$ .

The 3+1 representation of the affine connection  $\nabla$  on  $(\mathcal{M}, g)$  depends on the connection forms, which depend on the kinematic properties of the FIDO congruence according to the irreducible decomposition of  $\nabla \hat{n}$  in the sense of (4). The FIDO's kinematic properties are distinguished from kinematic properties of the fluid by a hat, i.e.  $\hat{\sigma}$ ,  $\hat{\omega}$ ,  $\hat{a}$  and  $\hat{\Theta}$  are the shear, vorticity, acceleration and expansion of the FIDO congruence, respectively. The fields  $\hat{\sigma}$ ,  $\hat{\omega}$  and  $\hat{a}$  live in  $(\Sigma_t, \boldsymbol{\gamma})$  and are therefore spatial tensor fields, which we denote by  $\hat{\boldsymbol{\sigma}} \equiv \mathbf{T}_{\hat{\sigma}}$ ,  $\hat{\boldsymbol{\omega}} \equiv \mathbf{T}_{\hat{\omega}}$  and  $\hat{\mathbf{a}} \equiv \mathbf{S}_{\hat{a}}$ . The induced metric  $\boldsymbol{\gamma}$  on  $(\Sigma_t, \boldsymbol{\gamma})$  is the horizontal part of the projector  $\gamma \equiv g + \hat{n} \otimes \hat{n}$  into the FIDO's frame.

The FIDO world lines are orthogonal to the hypersurfaces and thus rotationsfree,  $\hat{\omega} = \hat{\boldsymbol{\omega}} = 0$ . The FIDO's acceleration  $\hat{\mathbf{a}}$  is related to the lapse function by  $\hat{\mathbf{a}} = \boldsymbol{\nabla} \ln \alpha$ . The connection forms on  $(\mathcal{M}, g)$  can then be written in terms of the horizontal connection forms

on  $(\Sigma_t, \gamma)$ , and the horizontal parts of only two spatial tensor fields, namely

$$\hat{\mathbf{a}} = -\nabla \ln \alpha, \quad (61)$$

$$\mathbf{K} = \mathbf{T}_K. \quad (62)$$

Here  $\mathbf{K}$  is the extrinsic curvature tensor (second fundamental form), defined on  $(\mathcal{M}, g)$  by

$$\mathbf{K} \equiv -\nabla \hat{\mathbf{n}} = -\frac{1}{2} \mathcal{L}_{\hat{\mathbf{n}}} \gamma. \quad (63)$$

An equation for  $\mathbf{K}$  is obtained from the 3+1 representation of Einstein's equation (cf. (83) below). The 3+1 representation of the second equality of (63) is also recovered in the 3+1 formulation of Einstein's equations (cf. (82) below), and provides an equation for  $\gamma$ . This equation may be written in an alternative form based on the 3+1 representation of the first equality in (63), namely as

$$\mathbf{K} = -\left(\hat{\sigma} + \frac{1}{3} \hat{\Theta} \gamma + \frac{1}{2\alpha} \partial_t \gamma\right), \quad (64)$$

which confirms that  $\mathbf{K}$  is indeed spatial (recall that  $\hat{\omega} = 0$ ) and, furthermore, that  $\mathbf{K}$  is symmetric,  $\mathbf{K} = \langle \mathbf{K} \rangle^s$ .

For later use we give the 3+1 representation of  $\dot{D}A$ ,  $\dot{D}B$  and  $\dot{D}C$  according to (58)-(60),

$$\frac{E}{\dot{D}A} = \frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\beta} \right) \frac{E}{A} + \dot{D} \frac{E}{A}, \quad (65)$$

$$\frac{E}{\dot{D}B} = \frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\beta} \right) \frac{E}{B} + \dot{D} \frac{E}{B} - \mathbf{F} \cdot \mathbf{S}_B, \quad (66)$$

$$\frac{\mathbf{S}}{\dot{D}B} = \frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\beta} \right) \frac{\mathbf{S}}{B} + \dot{D} \frac{\mathbf{S}}{B} - \frac{E}{B} \mathbf{F} - \gamma \mathbf{K} \cdot \mathbf{S}_B, \quad (67)$$

$$\frac{E}{\dot{D}C} = \frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\beta} \right) \frac{E}{C} + \dot{D} \frac{E}{C} - 2\mathbf{F} \cdot \mathbf{S}_C, \quad (68)$$

$$\frac{\mathbf{S}}{\dot{D}C} = \frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\beta} \right) \frac{\mathbf{S}}{C} + \dot{D} \frac{\mathbf{S}}{C} - \gamma \mathbf{K} \cdot \mathbf{T}_C - \left( \frac{E}{C} \gamma + \mathbf{T}_C \right) \cdot \mathbf{F}, \quad (69)$$

$$\frac{\mathbf{T}}{\dot{D}C} = \frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\beta} \right) \frac{\mathbf{T}}{C} + \dot{D} \frac{\mathbf{T}}{C} - 2\gamma \mathbf{K} \cdot \mathbf{T}_C - \mathbf{F} \otimes \mathbf{S}_C - \mathbf{S}_C \otimes \mathbf{F}, \quad (70)$$

with the horizontal vector field  $\mathbf{F}$  defined on  $(\Sigma_t, \gamma)$  by

$$\mathbf{F} = -\gamma \nabla \ln \alpha + \mathbf{K} \cdot \mathbf{u}. \quad (71)$$

$\mathcal{L}_{\beta}$  is the Lie derivative on  $(\Sigma_t, \gamma)$  with respect to  $\beta$ . The horizontal projection operator  $\mathbf{h} = \gamma + \mathbf{u} \otimes \mathbf{u}$  defined on  $(\Sigma_t, \gamma)$  projects into the 2-space orthogonal to  $\mathbf{u}$ . Operators  $(\cdot)_{\mathbf{u}}$  and  $(\cdot)_{\mathbf{h}}$ ,  $\dot{D} \equiv \nabla_{\mathbf{u}} = \mathbf{u} \cdot \nabla$  and  $\mathbf{D} \equiv \nabla_{\mathbf{h}} = \mathbf{h} \cdot \nabla$  as well as  $\langle \cdot \rangle^s$  and  $\langle \cdot \rangle^a$  on  $(\Sigma_t, \gamma)$  are to be understood in analogy to the corresponding operators on  $(\mathcal{M}, g)$ . According to the

irreducible decomposition

$$\nabla \mathbf{u} = \boldsymbol{\sigma} + \boldsymbol{\omega} + \frac{1}{3}\vartheta \mathbf{h} - \mathbf{a} \otimes \mathbf{u}, \quad (72)$$

the kinematic properties of the fluid on  $(\Sigma_t, \gamma)$  are given by  $\boldsymbol{\sigma} \equiv \langle \nabla \mathbf{u} \rangle_{\mathbf{h}}^s$ ,  $\boldsymbol{\omega} \equiv \langle \nabla \mathbf{u} \rangle_{\mathbf{h}}^a$ ,  $\mathbf{a} \equiv \dot{\mathcal{D}}\mathbf{u}$  and  $\vartheta \equiv \nabla \cdot \mathbf{u}$ .

### 3.2 Conservation laws and Einstein's equations

The velocity  $\mathbf{u}$  has the 3+1 representation

$$\mathbf{u} = \gamma(\mathbf{e}_0 + \mathbf{v}) = \gamma \mathbf{e}_0 + \mathbf{u}, \quad (73)$$

where  $\gamma \equiv E_u = -\hat{\mathbf{n}} \cdot \mathbf{u}$  is the Lorentz factor with respect to the FIDOs and  $\mathbf{u} \equiv \mathbf{S}_u = \gamma \mathbf{v}$  is the horizontal part of  $\gamma \cdot \mathbf{u} = (g + \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) \cdot \mathbf{u} = \mathbf{u} + (\hat{\mathbf{n}} \cdot \mathbf{u})\hat{\mathbf{n}} = \mathbf{u} - \gamma\hat{\mathbf{n}}$ . Since  $\{\mathbf{e}_i\}$  is orthonormal, the  $v^i$  are *physical* 3-velocity components as measured by FIDOs. Similarly, the particle current  $\mathbf{n}$  has the 3+1 representation

$$\mathbf{n} = n\gamma(\mathbf{e}_0 + \mathbf{v}) = n(\gamma \mathbf{e}_0 + \mathbf{u}). \quad (74)$$

The stress energy tensor  $\mathbf{T}$  in (15) has the 3+1 representation

$$\frac{E}{\mathbf{T}} = (\rho + p + \Pi)\gamma^2 - (p + \Pi) + 2\gamma \frac{E}{\mathbf{q}} + \frac{E}{\pi}, \quad (75)$$

$$\frac{\mathbf{S}}{\mathbf{T}} = (\rho + p + \Pi)\gamma \mathbf{u} + \gamma \frac{\mathbf{S}}{\mathbf{q}} + \frac{E}{\mathbf{q}} \mathbf{u} + \frac{\mathbf{S}}{\pi}, \quad (76)$$

$$\frac{\mathbf{T}}{\mathbf{T}} = (\rho + p + \Pi)\mathbf{u} \otimes \mathbf{u} + (p + \Pi)\boldsymbol{\gamma} + \mathbf{u} \otimes \frac{\mathbf{S}}{\mathbf{q}} + \frac{\mathbf{S}}{\mathbf{q}} \otimes \mathbf{u} + \frac{\mathbf{T}}{\pi} \quad (77)$$

$$= \rho \mathbf{u} \otimes \mathbf{u} + (p + \Pi)\mathbf{h} + \mathbf{u} \otimes \frac{\mathbf{S}}{\mathbf{q}} + \frac{\mathbf{S}}{\mathbf{q}} \otimes \mathbf{u} + \frac{\mathbf{T}}{\pi}. \quad (78)$$

Note that  $\mathbf{q}$  and  $\pi$  are orthogonal to  $\mathbf{u}$  but not to  $\hat{\mathbf{n}}$ , and consequently  $\mathbf{q}$  and  $\pi$  are no spatial fields.

The 3+1 representation of particle number conservation (5), formally given by  $0 = E_{\nabla \cdot \mathbf{n}}$ , can be written as

$$0 = \frac{1}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) (\gamma n) + \frac{1}{\alpha} \nabla \cdot (\alpha n \mathbf{u}) - \gamma n \operatorname{tr}(\mathbf{K}). \quad (79)$$

The 3+1 representation of stress-energy conservation (6) splits into the energy equation,  $0 = E_{\nabla \cdot \mathbf{T}}$ , and the momentum equation,  $0 = \mathbf{S}_{\nabla \cdot \mathbf{T}}$ , which can be regarded as the spatial part of  $0 = \gamma \cdot (\nabla \cdot \mathbf{T})$ . Leaving  $\mathbf{T}$  unspecified, these are

$$0 = \frac{1}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \frac{E}{\mathbf{T}} - \operatorname{tr}(\mathbf{K}) \frac{E}{\mathbf{T}} + 2 \left( \frac{\mathbf{S}}{\mathbf{T}} \cdot \nabla \right) \ln \alpha + \nabla \cdot \frac{\mathbf{S}}{\mathbf{T}} - \operatorname{tr}(\mathbf{K} \cdot \frac{\mathbf{T}}{\mathbf{T}}), \quad (80)$$

$$0 = \frac{1}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \frac{\mathbf{S}}{\mathbf{T}} + \frac{E}{\mathbf{T}} \nabla \ln \alpha - 2 \mathbf{K} \cdot \frac{\mathbf{S}}{\mathbf{T}} - \operatorname{tr}(\mathbf{K}) \frac{\mathbf{S}}{\mathbf{T}} + \frac{1}{\alpha} \nabla \cdot (\alpha \frac{\mathbf{T}}{\mathbf{T}}). \quad (81)$$

For completeness we give the 3+1 representation of the Einstein equations (e.g. Durrer

& Straumann 1988). They consist of evolution equations for  $\boldsymbol{\gamma}$  and  $\mathbf{K}$ , obtained from the definition of the extrinsic curvature (63) and from the (space, space) components of (11). These dynamical equations are first order differential equations,

$$\frac{1}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \boldsymbol{\gamma} = -2\mathbf{K} , \quad (82)$$

$$\frac{1}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \mathbf{K} = \mathbf{Ri}(\boldsymbol{\gamma}) - 2\mathbf{K}^2 + \text{tr}(\mathbf{K})\mathbf{K} - 8\pi \frac{\mathbf{T}}{\mathbf{T}} + 4\pi \left( \text{tr}(\mathbf{K}) - \frac{\mathbf{E}}{\mathbf{T}} \right) \boldsymbol{\gamma} - \mathbf{He}(\ln \alpha) , \quad (83)$$

where  $\mathbf{K}^2 \equiv \mathbf{K} \cdot \mathbf{K}$ ,  $\mathbf{Ri}(\ )$  is the Ricci tensor and  $\mathbf{He}(\ )$  is the Hessian on  $(\Sigma_t, \boldsymbol{\gamma})$ , respectively. The (time, time) and (time, space) components of (11) yield the Hamiltonian and momentum constraints,

$$R - \text{tr}(\mathbf{K})^2 - \text{tr}(\mathbf{K}^2) = 16\pi \frac{E}{\mathbf{T}} , \quad (84)$$

$$\boldsymbol{\nabla} \cdot \mathbf{K} - \boldsymbol{\nabla} \text{tr}(\mathbf{K}) = 8\pi \frac{\mathbf{S}}{\mathbf{T}} , \quad (85)$$

where  $R$  is the curvature scalar on  $(\Sigma_t, \boldsymbol{\gamma})$ .

### 3.3 The standard constraint equations

The 3+1 representation of the constraint equations (23)-(25) for the thermodynamic fluxes  $\{\tilde{H}, \tilde{q}, \tilde{\pi}\}$  depends on the 3+1 representation of the kinematic properties  $\Theta, \mathbf{a}, \boldsymbol{\sigma}$  of the fluid, which can be calculated to

$$\frac{E}{\Theta} = W + \vartheta - \gamma \text{tr}(\mathbf{K}) , \quad (86)$$

$$\frac{E}{\mathbf{a}} = \gamma W + \dot{\mathcal{D}}\gamma - K_{\mathbf{u}} , \quad (87)$$

$$\frac{\mathbf{S}}{\mathbf{a}} = \gamma \mathbf{W} + \mathbf{a} , \quad (88)$$

$$\frac{E}{\boldsymbol{\sigma}} = \frac{1}{3} (\gamma^2 - 1) \left( 2W - \vartheta + \gamma \text{tr}(\mathbf{K}) \right) + \gamma \left( \dot{\mathcal{D}}\gamma - K_{\mathbf{u}} \right) , \quad (89)$$

$$\begin{aligned} \frac{\mathbf{S}}{\boldsymbol{\sigma}} &= \frac{1}{2} (\gamma^2 - 1) \mathbf{W} + \frac{\gamma}{6} \left( W - 2\vartheta + 2\gamma \text{tr}(\mathbf{K}) \right) \mathbf{u} + \frac{1}{2} \left( \mathbf{D}\gamma + \gamma \mathbf{a} \right) - \left( \frac{1}{2} K_{\mathbf{u}} \boldsymbol{\gamma} + \mathbf{K} \right) \cdot \mathbf{u} \\ &= \frac{1}{2} (\gamma^2 - 1) \mathbf{W} + \frac{1}{2} \left( W \mathbf{u} + \mathbf{D}\gamma + \gamma \mathbf{a} \right) - \frac{1}{3} \left( W + \vartheta - \gamma \text{tr}(\mathbf{K}) \right) \mathbf{u} - \left( \frac{1}{2} K_{\mathbf{u}} \boldsymbol{\gamma} + \mathbf{K} \right) \cdot \mathbf{u} \end{aligned} \quad (90)$$

$$\frac{\mathbf{T}}{\boldsymbol{\sigma}} = \frac{\gamma}{2} \left( \mathbf{W} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{W} \right) - \frac{1}{3} \left( W - \gamma \text{tr}(\mathbf{K}) \right) \mathbf{h} + \boldsymbol{\sigma} - \gamma \mathbf{K} , \quad (91)$$

with time derivatives contained in

$$W \equiv \frac{1}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \gamma + (\mathbf{u} \cdot \boldsymbol{\nabla}) \ln \alpha , \quad (92)$$

$$\mathbf{W} \equiv \frac{1}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \mathbf{u} - 2\mathbf{K} \cdot \mathbf{u} + \gamma \boldsymbol{\nabla} \ln \alpha . \quad (93)$$

Decomposing  $\mathbf{D} \ln T$  finally yields the 3+1 representation of the constraint equations

(23)-(25),

$$\frac{E}{\tilde{H}} = -\zeta \frac{E}{\Theta}, \quad (94)$$

$$\frac{E}{\tilde{q}} = -\lambda T \left[ (\gamma^2 - 1) \frac{1}{\alpha} (\partial_t - \mathcal{L}_{\boldsymbol{\beta}}) \ln T + \gamma \dot{\mathcal{D}} \ln T + \frac{E}{a} \right], \quad (95)$$

$$\frac{\mathbf{S}}{\tilde{q}} = -\lambda T \left[ \gamma \mathbf{u} \frac{1}{\alpha} (\partial_t - \mathcal{L}_{\boldsymbol{\beta}}) \ln T + \mathbf{D} \ln T + \frac{\mathbf{S}}{a} \right], \quad (96)$$

$$\frac{E}{\tilde{\pi}} = -2\eta \frac{E}{\sigma}, \quad (97)$$

$$\frac{\mathbf{S}}{\tilde{\pi}} = -2\eta \frac{\mathbf{S}}{\sigma}, \quad (98)$$

$$\frac{\mathbf{T}}{\tilde{\pi}} = -2\eta \frac{\mathbf{T}}{\sigma}. \quad (99)$$

### 3.4 The causal evolution equations

For the sake of simplicity the following discussion will be restricted to the causal evolution equations (44)-(46). Generalization to the full evolution equations (34)-(36) is straightforward. The required 3+1 representations of  $\dot{D}\Pi$ ,  $\dot{D}q$ ,  $\dot{D}\pi$  follow readily from (65)-(70). Remains to decompose products  $q \cdot a$  in (45) and  $\pi \cdot a$  in (46),

$$\frac{E}{q \cdot a} = \frac{E}{q} \frac{E}{a} + \frac{\mathbf{S}}{q} \cdot \frac{\mathbf{S}}{a}, \quad (100)$$

$$\frac{E}{\pi \cdot a} = \frac{E}{\pi} \frac{E}{a} + \frac{\mathbf{S}}{\pi} \cdot \frac{\mathbf{S}}{a}, \quad (101)$$

$$\frac{\mathbf{S}}{\pi \cdot a} = \frac{E}{a} \frac{\mathbf{S}}{\pi} + \frac{\mathbf{T}}{\pi} \cdot \frac{\mathbf{S}}{a}. \quad (102)$$

The 3+1 evolution equations for the causal thermodynamic fluxes  $\{E_{\Pi}, E_q, \mathbf{S}_q, E_{\pi}, \mathbf{S}_{\pi}, \mathbf{T}_{\pi}\}$  then follow from (44)-(46) to

$$\frac{\gamma}{\alpha} (\partial_t - \mathcal{L}_{\boldsymbol{\beta}}) \frac{E}{\tilde{H}} + \dot{\mathcal{D}} \frac{E}{\tilde{H}} = \frac{1}{\tau_0} \left( \frac{E}{\tilde{H}} - \frac{E}{\tilde{H}} \right), \quad (103)$$

$$\frac{\gamma}{\alpha} (\partial_t - \mathcal{L}_{\boldsymbol{\beta}}) \frac{E}{\tilde{q}} + \dot{\mathcal{D}} \frac{E}{\tilde{q}} = \frac{1}{\tau_1} \left( \frac{E}{\tilde{q}} - \frac{E}{\tilde{q}} \right) + \gamma \frac{E}{q \cdot a} + \mathbf{F} \cdot \frac{\mathbf{S}}{q}, \quad (104)$$

$$\frac{\gamma}{\alpha} (\partial_t - \mathcal{L}_{\boldsymbol{\beta}}) \frac{\mathbf{S}}{\tilde{q}} + \dot{\mathcal{D}} \frac{\mathbf{S}}{\tilde{q}} = \frac{1}{\tau_1} \left( \frac{\mathbf{S}}{\tilde{q}} - \frac{\mathbf{S}}{\tilde{q}} \right) + \frac{E}{q \cdot a} \mathbf{u} + \frac{E}{q} \mathbf{F} + \gamma \mathbf{K} \cdot \frac{\mathbf{S}}{q}, \quad (105)$$

$$\frac{\gamma}{\alpha} (\partial_t - \mathcal{L}_{\boldsymbol{\beta}}) \frac{E}{\tilde{\pi}} + \dot{\mathcal{D}} \frac{E}{\tilde{\pi}} = \frac{1}{\tau_2} \left( \frac{E}{\tilde{\pi}} - \frac{E}{\tilde{\pi}} \right) + 2\gamma \frac{E}{\pi \cdot a} + 2\mathbf{F} \cdot \frac{\mathbf{S}}{\pi}, \quad (106)$$

$$\frac{\gamma}{\alpha} (\partial_t - \mathcal{L}_{\boldsymbol{\beta}}) \frac{\mathbf{S}}{\tilde{\pi}} + \dot{\mathcal{D}} \frac{\mathbf{S}}{\tilde{\pi}} = \frac{1}{\tau_2} \left( \frac{\mathbf{S}}{\tilde{\pi}} - \frac{\mathbf{S}}{\tilde{\pi}} \right) + \gamma \frac{\mathbf{S}}{\pi \cdot a} + \frac{E}{\pi \cdot a} \mathbf{u} + \gamma \mathbf{K} \cdot \frac{\mathbf{S}}{\pi} + \left( \frac{E}{\pi} \boldsymbol{\gamma} + \frac{\mathbf{T}}{\pi} \right) \cdot \mathbf{F}, \quad (107)$$

$$\frac{\gamma}{\alpha} (\partial_t - \mathcal{L}_{\boldsymbol{\beta}}) \frac{\mathbf{T}}{\tilde{\pi}} + \dot{\mathcal{D}} \frac{\mathbf{T}}{\tilde{\pi}} = \frac{1}{\tau_2} \left( \frac{\mathbf{T}}{\tilde{\pi}} - \frac{\mathbf{T}}{\tilde{\pi}} \right) + \mathbf{u} \otimes \frac{\mathbf{S}}{\pi \cdot a} + \frac{\mathbf{S}}{\pi \cdot a} \otimes \mathbf{u} + 2\gamma \mathbf{K} \cdot \frac{\mathbf{T}}{\pi} + \mathbf{F} \otimes \frac{\mathbf{S}}{\pi} + \frac{\mathbf{S}}{\pi} \otimes \mathbf{F}, \quad (108)$$

with  $\mathbf{F}$  according to (71).

### 3.5 The weakly dissipative limit

The limit of weak dissipation discussed in section 2.6 implies three simplifications of the 3+1 equations of causal hydrodynamics. The first simplification concerns the 3+1 conservation laws (80) and (81) for energy and momentum, where dissipative contributions  $\{\Pi, \mathbf{q}, \boldsymbol{\pi}\}$  to the matter sources  $\{E_{\text{T}}, \mathbf{S}_{\text{T}}, \mathbf{T}_{\text{T}}\}$  in (75)-(78) can be dropped wherever they enter algebraically, and only temporal and spatial gradients remain. The second simplification affects the 3+1 representation of the kinematic properties of the fluid, (86)-(91). Using the geodesic conditions  $E_{\text{a}} = \mathbf{S}_{\text{a}} = 0$ , (87) and (88) yield simplified expressions for  $W$  and  $\mathbf{W}$ ,

$$W = -\dot{\mathcal{D}} \ln \gamma + K_{\mathbf{u}}/\gamma, \quad (109)$$

$$\mathbf{W} = -\mathbf{a}/\gamma, \quad (110)$$

which are no longer time-dependent. Therefore, any kinematic properties (86)-(91) simplify to time-independent expressions

$$\underline{\varrho}E = \vartheta - \dot{\mathcal{D}} \ln \gamma + K_{\mathbf{u}}/\gamma - \gamma \text{tr}(\mathbf{K}), \quad (111)$$

$$\underline{\text{a}}E = 0, \quad (112)$$

$$\underline{\text{a}}\mathbf{S} = 0, \quad (113)$$

$$\begin{aligned} \underline{\varrho}E &= \frac{1}{3}(\gamma^2 + 2) \left( \dot{\mathcal{D}} \ln \gamma - K_{\mathbf{u}}/\gamma \right) - \frac{1}{3}(\gamma^2 - 1) \left( \vartheta - \gamma \text{tr}(\mathbf{K}) \right) \\ &= \frac{2}{3} \left( \dot{\mathcal{D}} \ln \gamma - K_{\mathbf{u}}/\gamma \right) + \frac{1}{3} \left( \vartheta - \gamma \text{tr}(\mathbf{K}) \right) + \frac{1}{3} \left( \dot{\mathcal{D}} \ln \gamma - K_{\mathbf{u}}/\gamma - \vartheta + \gamma \text{tr}(\mathbf{K}) \right) \gamma^2, \end{aligned} \quad (114)$$

$$\underline{\varrho}\mathbf{S} = \frac{1}{2} \left( \mathbf{a}/\gamma + \boldsymbol{\nabla} \gamma \right) + \mathbf{K} \cdot \mathbf{u} + \frac{1}{3} \left( \dot{\mathcal{D}} \ln \gamma - K_{\mathbf{u}}/\gamma - \vartheta + \gamma \text{tr}(\mathbf{K}) \right) \gamma \mathbf{u}, \quad (115)$$

$$\underline{\varrho}\mathbf{T} = \underline{\boldsymbol{\sigma}} - \gamma \mathbf{K} + \frac{1}{3} \left( \dot{\mathcal{D}} \ln \gamma - K_{\mathbf{u}}/\gamma + \gamma \text{tr}(\mathbf{K}) \right) \mathbf{h}, \quad (116)$$

with  $\underline{\boldsymbol{\sigma}} = \langle \boldsymbol{\nabla} \mathbf{u} \rangle^{\text{s}} - 1/2 \vartheta \mathbf{h}$ . Finally, the weak dissipation limit affects the 3+1 evolution equations (103)-(108) for the thermodynamic fluxes  $\{E_{\text{II}}, E_{\text{q}}, \mathbf{S}_{\text{q}}, E_{\text{π}}, \mathbf{S}_{\text{π}}, \mathbf{T}_{\text{π}}\}$ , where products  $E_{\text{q}\cdot\text{a}}, E_{\text{π}\cdot\text{a}}, \mathbf{S}_{\text{π}\cdot\text{a}}$  as in (100)-(102) are dropped. This yields the weak dissipative evolution equations for the thermodynamic fluxes  $\{E_{\text{II}}, E_{\text{q}}, \mathbf{S}_{\text{q}}, E_{\text{π}}, \mathbf{S}_{\text{π}}, \mathbf{T}_{\text{π}}\}$ ,

$$\frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \underline{\underline{E}} + \dot{\mathcal{D}} \underline{\underline{E}} = \frac{1}{\tau_0} \left( \underline{\underline{E}} - \underline{\underline{E}} \right), \quad (117)$$

$$\frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \underline{\underline{E}} + \dot{\mathcal{D}} \underline{\underline{E}} = \frac{1}{\tau_1} \left( \underline{\underline{E}} - \underline{\underline{E}} \right) + \mathbf{F} \cdot \underline{\underline{\mathbf{S}}}, \quad (118)$$

$$\frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \underline{\underline{\mathbf{S}}} + \dot{\mathcal{D}} \underline{\underline{\mathbf{S}}} = \frac{1}{\tau_1} \left( \underline{\underline{\mathbf{S}}} - \underline{\underline{\mathbf{S}}} \right) + \underline{\underline{E}} \mathbf{F} + \gamma \mathbf{K} \cdot \underline{\underline{\mathbf{S}}}, \quad (119)$$

$$\frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\boldsymbol{\beta}} \right) \underline{\underline{E}} + \dot{\mathcal{D}} \underline{\underline{E}} = \frac{1}{\tau_2} \left( \underline{\underline{E}} - \underline{\underline{E}} \right) + 2\mathbf{F} \cdot \underline{\underline{\mathbf{S}}}, \quad (120)$$



$$\frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\beta} \right) \mathbf{S}_{\underline{\pi}} + \dot{\mathbf{S}}_{\underline{\pi}} = \frac{1}{\tau_2} \left( \mathbf{S}_{\underline{\pi}} - \mathbf{S}_{\underline{\pi}} \right) + \gamma \mathbf{K} \cdot \mathbf{S}_{\underline{\pi}} + \left( \frac{E}{\underline{\pi}} \boldsymbol{\gamma} + \frac{\mathbf{T}}{\underline{\pi}} \right) \cdot \mathbf{F} , \quad (121)$$

$$\frac{\gamma}{\alpha} \left( \partial_t - \mathcal{L}_{\beta} \right) \mathbf{T}_{\underline{\pi}} + \dot{\mathbf{T}}_{\underline{\pi}} = \frac{1}{\tau_2} \left[ \mathbf{T}_{\underline{\pi}} - \mathbf{T}_{\underline{\pi}} \right] + 2\gamma \mathbf{K} \cdot \mathbf{T}_{\underline{\pi}} + \mathbf{F} \otimes \mathbf{S}_{\underline{\pi}} + \mathbf{S}_{\underline{\pi}} \otimes \mathbf{F} . \quad (122)$$

## 4 STATIONARY AND AXISYMMETRIC BACKGROUND SPACETIMES

The 3+1 equations of dissipative hydrodynamics are specified to the class of stationary, axisymmetric background spacetimes. This situation is realized if the fluid under consideration has negligible influence on the gravitational field of a central object, and in addition this field is known to be stationary and axisymmetric. These assumptions include most applications related to accretion/ejection flows in the vicinity of compact objects. For rotating black holes the vacuum metric is Kerr, and we give the equations also for this special case.

### 4.1 Implications of symmetries

The general form of a stationary, axisymmetric vacuum spacetime can be put in a form which is symmetric under a simultaneous change of sign of  $t$  and  $\phi$ , the Killing coordinates associated with the commuting time and axial Killing vector fields  $\mathbf{k}$  and  $\mathbf{m}$ . Choosing the remaining two meridional coordinates as spherical coordinates allows to write  $g$  as

$$g = -\alpha^2 dt \otimes dt + \tilde{\omega}^2 (d\phi - \omega dt) \otimes (d\phi - \omega dt) + e^{2\mu} dr \otimes dr + e^{2\nu} d\theta \otimes d\theta , \quad (123)$$

with the invariant metric coefficients

$$\tilde{\omega}^2 = m^2 , \quad \omega = -\mathbf{k} \cdot \mathbf{m} / m^2 , \quad \alpha^2 = -k^2 + \mathbf{k} \cdot \mathbf{m} / m^2 . \quad (124)$$

For the physical interpretation of  $\tilde{\omega}$ ,  $\omega$  and  $\alpha$  see e.g. Bardeen (1970). The generic choice of the fiducial congruence is (see e.g. Thorne & Macdonald (1982) for criteria that uniquely fix this choice)

$$\hat{\mathbf{n}} = \mathbf{e}_0 = \frac{1}{\alpha} (\mathbf{k} + \omega \mathbf{m}) . \quad (125)$$

These FIDOs possess vanishing specific angular momentum,  $\hat{n}_\phi = \hat{\mathbf{n}} \cdot \mathbf{m} = 0$ . Therefore they correspond to Bardeen's (1970) *zero angular momentum observers* (ZAMOs). Furthermore, since  $\hat{\mathbf{n}} \cdot \mathbf{m} = 0$ ,  $\mathbf{m}$  is a spatial vector, which we denote on  $(\Sigma_t, \boldsymbol{\gamma})$  by  $\mathbf{m} \equiv \mathbf{S}_m$ . Comparison with (57) shows that the shift vector has to be chosen as

$$\boldsymbol{\beta} = -\omega \mathbf{m} , \quad (126)$$

and the metric induced on  $(\Sigma_t, \gamma)$  becomes

$$\gamma = \tilde{\omega}^2 d\phi \otimes d\phi + e^{2\mu} dr \otimes dr + e^{2\nu} d\theta \otimes d\theta . \quad (127)$$

The 3+1 representation of Killing's equation for  $\mathbf{k}$  and  $\mathbf{m}$ , together with their commutivity, allows to establish the following relations (Thorne & Macdonald 1982)

$$\partial_t \mu = 0 , \quad \partial_t \nu = 0 , \quad \partial_t \omega = 0 , \quad \partial_t \mathbf{m} = 0 , \quad \partial_t \gamma = 0 , \quad (128)$$

$$\mathbf{m} \cdot \nabla \mu = 0 , \quad \mathbf{m} \cdot \nabla \nu = 0 , \quad \mathbf{m} \cdot \nabla \omega = 0 , \quad \langle \nabla \mathbf{m} \rangle^s = 0 , \quad (129)$$

$$\hat{\Theta} = 0 , \quad \hat{\sigma} = \frac{1}{2\alpha} (\mathbf{m} \otimes \nabla \omega + \nabla \omega \otimes \mathbf{m}) . \quad (130)$$

Note that  $\langle \nabla \mathbf{m} \rangle^s = 0$  states that  $\mathbf{m}$  is a Killing vector field on  $(\Sigma_t, \gamma)$ . However, as a consequence of  $\nabla \omega \neq 0$ ,  $\beta$  is not a Killing vector field on  $(\Sigma_t, \gamma)$ . As a consequence of  $\partial_t \gamma$  and  $\hat{\Theta} = 0$ , the horizontal field  $\mathbf{K}$  according to (64) reduces to

$$\mathbf{K} = -\hat{\sigma} = -\frac{1}{2\alpha} (\mathbf{m} \otimes \nabla \omega + \nabla \omega \otimes \mathbf{m}) . \quad (131)$$

Therefore  $\mathbf{K}$  measures the shear of hypersurfaces  $\Sigma_t$ , which vanishes for  $\omega = 0$ .

## 4.2 Gravitomagnetic and gravitoelectric tensor fields

A characteristic phenomenon in axisymmetric spacetimes is the dragging of inertial frames. Physical implications due to this effect (see Thorne et al. 1986 for the case of the Kerr metric) are often described in terms of the gravitomagnetic tensor field  $\mathbf{H}$ , defined on  $(\Sigma_t, \gamma)$  by

$$\mathbf{H} \equiv \frac{1}{\alpha} \nabla \beta = -\frac{1}{\alpha} \nabla (\omega \mathbf{m}) = -\frac{1}{\alpha} [\omega \nabla \mathbf{m} + \nabla \omega \otimes \mathbf{m}] . \quad (132)$$

It is clear from (131) that  $\mathbf{K} = \langle \mathbf{H} \rangle^s$ , and therefore

$$\mathbf{K} \cdot \mathbf{T} = \frac{1}{2} (\mathbf{H} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{H}) . \quad (133)$$

The antisymmetric part of  $\mathbf{H}$  can be expressed as an axial vector field on  $(\Sigma_t, \gamma)$ , namely the gravitomagnetic vector field

$$\mathbf{J} \equiv \frac{1}{\alpha} \nabla \wedge \beta . \quad (134)$$

For an arbitrary vector field  $\mathbf{S}$  on  $(\Sigma_t, \gamma)$  one finds the relation

$$\mathbf{H} \cdot \mathbf{S} = \mathbf{K} \cdot \mathbf{S} - \frac{1}{2} \mathbf{J} \wedge \mathbf{S} . \quad (135)$$

The Lie derivatives  $\mathcal{L}_\beta$  in the 3+1 equations derived in section 3 can then be expressed as

$$\mathcal{L}_\beta E = (\beta \cdot \nabla) E , \quad (136)$$

$$\mathcal{L}_\beta \mathbf{S} = (\beta \cdot \nabla) \mathbf{S} - (\mathbf{S} \cdot \nabla) \beta = (\beta \cdot \nabla) \mathbf{S} - \alpha \mathbf{S} \cdot \mathbf{H} , \quad (137)$$

$$\mathcal{L}_{\boldsymbol{\beta}} \mathbf{T} = (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}) \mathbf{T} - 2\langle (\mathbf{T} \cdot \boldsymbol{\nabla}) \boldsymbol{\beta} \rangle^s = (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}) \mathbf{T} - 2\alpha \langle \mathbf{T} \cdot \mathbf{H} \rangle^s = (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}) \mathbf{T} - 2\mathbf{T} \cdot \mathbf{K}. \quad (138)$$

The term  $\mathbf{S} \cdot \mathbf{H}$  in (137) may be expressed by either  $\mathbf{H}$  as in (132) or in terms of  $\mathbf{J}$  and  $\mathbf{K}$  via (135). In addition to the gravitomagnetic tensor fields  $\mathbf{H}$  and  $\mathbf{J}$ , we introduce the gravitoelectric vector field  $\mathbf{G}$ , defined on  $(\Sigma_t, \boldsymbol{\gamma})$  by

$$\mathbf{G} \equiv -\hat{\mathbf{a}} = -\boldsymbol{\nabla} \ln \alpha. \quad (139)$$

This field measures the gravitational acceleration measured by the fiducial observers.

### 4.3 Conservation laws

The 3+1 conservation laws for particle number (79), energy (80) and momentum (81) in a stationary, axisymmetric background read

$$0 = \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) (\gamma n) + \frac{1}{\alpha} \boldsymbol{\nabla} (\alpha n \mathbf{u}), \quad (140)$$

$$0 = \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \frac{E}{\mathbf{T}} + \boldsymbol{\nabla} \cdot \frac{\mathbf{S}}{\mathbf{T}} - 2 \frac{\mathbf{S}}{\mathbf{T}} \cdot \mathbf{G} - \text{tr}(\mathbf{K} \cdot \frac{\mathbf{T}}{\mathbf{T}}), \quad (141)$$

$$\begin{aligned} 0 &= \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \frac{\mathbf{S}}{\mathbf{T}} - \frac{E}{\mathbf{T}} \mathbf{G} - \mathbf{H} \cdot \frac{\mathbf{S}}{\mathbf{T}} + \frac{1}{\alpha} \boldsymbol{\nabla} (\alpha \frac{\mathbf{T}}{\mathbf{T}}) \\ &= \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \frac{\mathbf{S}}{\mathbf{T}} + \boldsymbol{\nabla} \cdot \frac{\mathbf{T}}{\mathbf{T}} - \mathbf{H} \cdot \frac{\mathbf{S}}{\mathbf{T}} - \left( \frac{E}{\mathbf{T}} \boldsymbol{\gamma} + \frac{\mathbf{T}}{\mathbf{T}} \right) \cdot \mathbf{G}. \end{aligned} \quad (142)$$

### 4.4 Evolution equations

The 3+1 representation (86)-(91) of the kinematic properties of the fluid in stationary, axisymmetric background read

$$\frac{E}{\vartheta} = W + \vartheta, \quad (143)$$

$$\frac{E}{\mathbf{a}} = \gamma W + \dot{\mathcal{D}}\gamma + \hat{\boldsymbol{\sigma}}_{\mathbf{u}}, \quad (144)$$

$$\frac{\mathbf{S}}{\mathbf{a}} = \gamma \mathbf{W} + \mathbf{a}, \quad (145)$$

$$\frac{E}{\sigma} = \frac{3}{2} (\gamma^2 - 1) (W - 2\vartheta) + \gamma (\dot{\mathcal{D}}\gamma + \hat{\boldsymbol{\sigma}}_{\mathbf{u}}), \quad (146)$$

$$\frac{\mathbf{S}}{\sigma} = \frac{1}{2} (\gamma^2 - 1) \mathbf{W} + \frac{\gamma}{6} (W - 2\vartheta) \mathbf{u} + \frac{1}{2} (\mathbf{D}\gamma + \gamma \mathbf{a}) - (\hat{\boldsymbol{\sigma}}_{\mathbf{u}} \boldsymbol{\gamma} + \hat{\boldsymbol{\sigma}}) \cdot \mathbf{u}, \quad (147)$$

$$\frac{\mathbf{T}}{\sigma} = \frac{\gamma}{2} (\mathbf{W} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{W}) - \frac{1}{3} W \mathbf{h} + \boldsymbol{\sigma} + \gamma \hat{\boldsymbol{\sigma}} \quad (148)$$

with time derivatives contained in

$$W = \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \gamma - G_{\mathbf{u}}, \quad (149)$$

$$\mathbf{W} = \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \mathbf{u} - \mathbf{H} \cdot \mathbf{u} - \gamma \mathbf{G}. \quad (150)$$

The 3+1 representation of the standard constraint equations (94)-(99) can be written as

$$\frac{E}{\tilde{H}} = -\zeta \frac{E}{\Theta} , \quad (151)$$

$$\frac{E}{\tilde{q}} = -\kappa T \left[ (\gamma^2 - 1) \frac{1}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \ln T + \gamma \dot{\mathcal{D}} \ln T + \frac{E}{\tilde{a}} \right] , \quad (152)$$

$$\frac{\mathbf{S}}{\tilde{q}} = -\kappa T \left[ \gamma \frac{\mathbf{u}}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \ln T + \mathbf{D} \ln T + \frac{\mathbf{S}}{\tilde{a}} \right] , \quad (153)$$

$$\frac{E}{\tilde{\pi}} = -2\eta \frac{E}{\sigma} , \quad (154)$$

$$\frac{\mathbf{S}}{\tilde{\pi}} = -2\eta \frac{\mathbf{S}}{\sigma} , \quad (155)$$

$$\frac{\mathbf{T}}{\tilde{\pi}} = -2\eta \frac{\mathbf{T}}{\sigma} . \quad (156)$$

The 3+1 representation of the causal evolution equations (103)-(108) reads

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \frac{E}{\tilde{H}} + \dot{\mathcal{D}} \frac{E}{\tilde{H}} = \frac{1}{\tau_0} \left( \frac{E}{\tilde{H}} - \frac{E}{\tilde{H}} \right) , \quad (157)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \frac{E}{\tilde{q}} + \dot{\mathcal{D}} \frac{E}{\tilde{q}} = \frac{1}{\tau_1} \left( \frac{E}{\tilde{q}} - \frac{E}{\tilde{q}} \right) + \gamma \frac{E}{\tilde{q} \cdot \tilde{a}} + \mathbf{F} \cdot \frac{\mathbf{S}}{\tilde{q}} , \quad (158)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \frac{\mathbf{S}}{\tilde{q}} + \dot{\mathcal{D}} \frac{\mathbf{S}}{\tilde{q}} = \frac{1}{\tau_1} \left( \frac{\mathbf{S}}{\tilde{q}} - \frac{\mathbf{S}}{\tilde{q}} \right) + \frac{E}{\tilde{q} \cdot \tilde{a}} \mathbf{u} + \frac{E}{\tilde{q}} \mathbf{F} + \frac{1}{2} \gamma \mathbf{J} \wedge \frac{\mathbf{S}}{\tilde{q}} , \quad (159)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \frac{E}{\tilde{\pi}} + \dot{\mathcal{D}} \frac{E}{\tilde{\pi}} = \frac{1}{\tau_2} \left( \frac{E}{\tilde{\pi}} - \frac{E}{\tilde{\pi}} \right) + 2\gamma \frac{E}{\tilde{\pi} \cdot \tilde{a}} + 2\gamma \mathbf{F} \cdot \frac{\mathbf{S}}{\tilde{\pi}} , \quad (160)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \frac{\mathbf{S}}{\tilde{\pi}} + \dot{\mathcal{D}} \frac{\mathbf{S}}{\tilde{\pi}} = \frac{1}{\tau_2} \left( \frac{\mathbf{S}}{\tilde{\pi}} - \frac{\mathbf{S}}{\tilde{\pi}} \right) + \gamma \frac{\mathbf{S}}{\tilde{\pi} \cdot \tilde{a}} + \frac{E}{\tilde{\pi} \cdot \tilde{a}} \mathbf{u} + \frac{1}{2} \gamma \mathbf{J} \wedge \frac{\mathbf{S}}{\tilde{\pi}} + \left( \frac{E}{\tilde{\pi}} \boldsymbol{\gamma} + \frac{\mathbf{T}}{\tilde{\pi}} \right) \cdot \mathbf{F} , \quad (161)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} \right) \frac{\mathbf{T}}{\tilde{\pi}} + \dot{\mathcal{D}} \frac{\mathbf{T}}{\tilde{\pi}} = \frac{1}{\tau_2} \left( \frac{\mathbf{T}}{\tilde{\pi}} - \frac{\mathbf{T}}{\tilde{\pi}} \right) + \mathbf{u} \otimes \frac{\mathbf{S}}{\tilde{\pi} \cdot \tilde{a}} + \frac{\mathbf{S}}{\tilde{\pi} \cdot \tilde{a}} \otimes \mathbf{u} + \gamma \mathbf{F} \otimes \frac{\mathbf{S}}{\tilde{\pi}} + \gamma \frac{\mathbf{S}}{\tilde{\pi}} \otimes \mathbf{F} , \quad (162)$$

with

$$\mathbf{F} = \gamma \mathbf{G} - \hat{\boldsymbol{\sigma}} \cdot \mathbf{u} . \quad (163)$$

#### 4.5 Weakly dissipative limit

The 3+1 representation of the kinematic properties of the fluid in the weak dissipation limit can be written as

$$\underline{\varrho} E = \vartheta - \dot{\mathcal{D}} \ln \gamma + \hat{\boldsymbol{\sigma}}_{\mathbf{u}} / \gamma , \quad (164)$$

$$\underline{a} E = 0 , \quad (165)$$

$$\underline{a} \mathbf{S} = 0 , \quad (166)$$

$$\begin{aligned} \underline{\sigma} E &= \frac{1}{3} (\gamma^2 + 2) \left( \dot{\mathcal{D}} \ln \gamma - \hat{\boldsymbol{\sigma}}_{\mathbf{u}} / \gamma \right) - \frac{1}{3} (\gamma^2 - 1) \vartheta \\ &= \frac{2}{3} \left( \dot{\mathcal{D}} \ln \gamma - \hat{\boldsymbol{\sigma}}_{\mathbf{u}} / \gamma \right) + \frac{1}{3} \vartheta + \frac{1}{3} \left( \dot{\mathcal{D}} \ln \gamma - \hat{\boldsymbol{\sigma}}_{\mathbf{u}} / \gamma - \vartheta \right) \gamma^2 , \end{aligned} \quad (167)$$

$$\underline{\mathbf{S}} = \frac{1}{2}(\mathbf{a}/\gamma + \nabla\gamma) + \hat{\mathbf{\sigma}} \cdot \mathbf{u} + \frac{1}{3}(\dot{\mathcal{D}} \ln \gamma - \hat{\mathbf{\sigma}}_{\mathbf{u}}/\gamma - \vartheta)\gamma\mathbf{u}, \quad (168)$$

$$\underline{\mathbf{T}} = \underline{\mathbf{\sigma}} - \gamma\hat{\mathbf{\sigma}} + \frac{1}{3}(\dot{\mathcal{D}} \ln \gamma - \hat{\mathbf{\sigma}}_{\mathbf{u}}/\gamma)\mathbf{h}. \quad (169)$$

The 3+1 weak dissipation evolution equations (103)-(108) for  $\{E_{\underline{\mathbb{H}}}, E_{\underline{\mathbb{Q}}}, \mathbf{S}_{\underline{\mathbb{Q}}}, E_{\underline{\mathbb{T}}}, \mathbf{S}_{\underline{\mathbb{T}}}, \mathbf{T}_{\underline{\mathbb{T}}}\}$  can be written as

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \nabla \right) \frac{E}{\underline{\mathbb{H}}} + \dot{\mathcal{D}} \frac{E}{\underline{\mathbb{H}}} = \frac{1}{\tau_0} \left( \frac{E}{\underline{\mathbb{H}}} - \frac{E}{\underline{\mathbb{H}}} \right), \quad (170)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \nabla \right) \frac{E}{\underline{\mathbb{Q}}} + \dot{\mathcal{D}} \frac{E}{\underline{\mathbb{Q}}} = \frac{1}{\tau_1} \left( \frac{E}{\underline{\mathbb{Q}}} - \frac{E}{\underline{\mathbb{Q}}} \right) + \mathbf{F} \cdot \mathbf{S}_{\underline{\mathbb{Q}}}, \quad (171)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \nabla \right) \mathbf{S}_{\underline{\mathbb{Q}}} + \dot{\mathcal{D}} \mathbf{S}_{\underline{\mathbb{Q}}} = \frac{1}{\tau_1} \left( \mathbf{S}_{\underline{\mathbb{Q}}} - \mathbf{S}_{\underline{\mathbb{Q}}} \right) + \frac{E}{\underline{\mathbb{Q}}} \mathbf{F} + \frac{1}{2} \gamma \mathbf{J} \wedge \mathbf{S}_{\underline{\mathbb{Q}}}, \quad (172)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \nabla \right) \frac{E}{\underline{\mathbb{T}}} + \dot{\mathcal{D}} \frac{E}{\underline{\mathbb{T}}} = \frac{1}{\tau_2} \left( \frac{E}{\underline{\mathbb{T}}} - \frac{E}{\underline{\mathbb{T}}} \right) + 2 \mathbf{F} \cdot \mathbf{S}_{\underline{\mathbb{T}}}, \quad (173)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \nabla \right) \mathbf{S}_{\underline{\mathbb{T}}} + \dot{\mathcal{D}} \mathbf{S}_{\underline{\mathbb{T}}} = \frac{1}{\tau_2} \left( \mathbf{S}_{\underline{\mathbb{T}}} - \mathbf{S}_{\underline{\mathbb{T}}} \right) + \frac{1}{2} \gamma \mathbf{J} \wedge \mathbf{S}_{\underline{\mathbb{T}}} + \left( \frac{E}{\underline{\mathbb{T}}} \boldsymbol{\gamma} + \mathbf{T}_{\underline{\mathbb{T}}} \right) \cdot \mathbf{F}, \quad (174)$$

$$\frac{\gamma}{\alpha} \left( \frac{\partial}{\partial t} - \boldsymbol{\beta} \cdot \nabla \right) \mathbf{T}_{\underline{\mathbb{T}}} + \dot{\mathcal{D}} \mathbf{T}_{\underline{\mathbb{T}}} = \frac{1}{\tau_2} \left( \mathbf{T}_{\underline{\mathbb{T}}} - \mathbf{T}_{\underline{\mathbb{T}}} \right) + \mathbf{F} \otimes \mathbf{S}_{\underline{\mathbb{T}}} + \mathbf{S}_{\underline{\mathbb{T}}} \otimes \mathbf{F} \quad (175)$$

with  $\mathbf{F}$  according to (163).

#### 4.6 Specification to Kerr geometry

The behaviour of fluids in the vicinity of a rotating black hole is governed by the conservation laws and evolution equations in Kerr metric. The Kerr metric is a two parameter family of stationary, axisymmetric vacuum spacetimes. Parameters are the mass  $M$  and the specific angular momentum  $a \equiv J/M$  of the black hole. In terms of the functions

$$\Delta \equiv r^2 + a^2 - 2Mr, \quad \varrho^2 \equiv r^2 + a^2 \cos^2 \theta, \quad \Sigma^2 \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \quad (176)$$

the lapse function  $\alpha$  and the non-vanishing components of the shift vector  $\boldsymbol{\beta}$  and the metric  $\boldsymbol{\gamma}$  according to (56) are

$$\alpha = \frac{\varrho \sqrt{\Delta}}{\Sigma}, \quad \beta^\phi = -\omega = -\frac{2aMr}{\Sigma^2}, \quad \tilde{\omega} = \frac{\Sigma}{\varrho} \sin \theta, \quad e^{2\mu} = \frac{\varrho^2}{\Delta}, \quad e^{2\nu} = \varrho^2. \quad (177)$$

A consistent treatment of viscous hydrodynamics in the vicinity of a rotating black hole would require a careful analysis of the boundary conditions at the horizon. Such an analysis is best performed within the concept of a *stretched horizon*, defined by a small value of lapse  $\alpha$ . We refer to the *membrane paradigm* of Thorne et al. (1986), where this approach is applied to ideal magnetohydrodynamics and Maxwell's equations. This work also contains a coordinate representation of  $\mathbf{G}$  and  $\mathbf{H}$ .

#### 4.7 Stationary and axisymmetric flows

The assumptions of stationarity and axisymmetry so far concerned exclusively the background metric. In many problems it is justified to assume these symmetries to hold also for the fluid configuration under consideration. The equations for stationary flow are obtained by setting  $\partial_t = 0$  in the 3+1 equations, since now  $\mathbf{k}$  is a Killing vector field also for the fluid. Similarly, the equations for axisymmetric flow follow from setting  $\boldsymbol{\beta} \cdot \boldsymbol{\nabla} = 0$ , since  $\mathbf{m}$  then is a Killing vector field also for the fluid.

The post-Newtonian limit of causal viscous hydrodynamics is recovered from the 3+1 equations by setting  $\alpha = 1$  ( $\mapsto \mathbf{G} = 0$ ) and  $\boldsymbol{\beta} = 0$  ( $\mapsto \mathbf{H} = \hat{\boldsymbol{\sigma}} = 0$ ), and furthermore setting  $\gamma$  equal to unity, while gradients of  $\gamma$  are retained. This limit contains still non-vanishing quantities of post-Newtonian order  $\nabla_{\mathbf{u}}\mathbf{u}$ . Neglecting these then yields the Newtonian limit of causal viscous hydrodynamics, where  $E_\Theta$ ,  $E_a$ ,  $E_\sigma$ ,  $\mathbf{S}_\sigma$  vanish. A related causal description of viscous angular momentum transfer in Newtonian accretion disc boundary layers was considered by Papaloizou & Szuskiwicz (1994) and Kley & Papaloizou (1997).

## 5 CONCLUSIONS

We have provided a complete set of equations for dissipative relativistic hydrodynamics in their 3+1 representation. Furthermore, we have specified the general system to the class of stationary axisymmetric vacuum spacetimes, with the Kerr metric as the most relevant astrophysical example. For this case we have written the equations in a form where the dragging of inertial frames is described by the gravitomagnetic tensor field. This allows to combine the equations with the 3+1 Maxwell's equations as in Thorne & Macdonald (1982).

Causality has been accounted for by using the extended causal description of thermodynamics, relativistically formulated on a phenomenological level by Israel (1976) and Stewart (1977) and justified by kinetic theory (Israel & Stewart 1979). In contrast to the conventionally used compressible Navier-Stokes description of non-ideal hydrodynamics, the equations of extended causal thermodynamics guarantee finite propagation speeds of heat and viscous signals and yield stable local thermodynamic equilibria (Hiscock & Lindblom 1983).

A causality preserving formulation is required whenever the thermodynamic timescale becomes comparable to the dynamical timescale and therefore the assumption of local thermodynamic equilibrium is not justified. This is particularly the case in supersonic flows and/or processes in the vicinity of the event horizon. Astrophysical examples and thus po-

tential fields for application of causal thermodynamics in the formulation presented here include the gravitational collapse of stars (e.g. Baumgarte et al. 1995), the innermost parts of accretion discs around black holes (e.g. Peitz & Appl 1997) or the collision of neutron stars (e.g. Rasio & Shapiro 1992). In many problems of interest the inertia due to the dissipative contributions to the stress-energy tensor can be neglected. The 3+1 formulation of the corresponding simplified set of equations is given.

The five conservation laws for particle number, energy and momentum, the ten Einstein equations for the metric tensor and the ten evolution equations for the thermodynamic fluxes form a hyperbolic system of first order PDEs tractable by numerical methods (e.g. Bonazzola et al. (1993)).

We are grateful to M. Camenzind, R. Khanna and S. Spindeldreher for enlightening discussions. Support by Deutsche Forschungsgemeinschaft under SFB 328 (JP) and SFB 359 (SA), and by the French Ministry of Foreign Affairs (SA) is acknowledged.

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